Noise Can Reduce Disorder in Chaotic Dynamics

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Abstract – We evoke the idea of representation of the chaotic attractor by the set of unstable periodic orbits and disclose a novel noise-induced ordering phenomenon. For long unstable periodic orbits forming the strange attractor the weights (or natural measure) is generally highly inhomogeneous over the set, either diminishing or enhancing the contribution of these orbits into system dynamics. We show analytically and numerically a weak noise to reduce this inhomogeneity and, additionally to obvious perturbing impact, make a regularizing influence on the chaotic dynamics. This universal effect is rooted into the nature of deterministic chaos.

Introduction. – During few recent decades, noise was found to have ability not only to make an obvious disordering impact, but also to play a constructive role, increasing degree of order in system dynamics [1–7]. Most remarkable advances are the phenomena of stochastic [1] and coherence resonances [2]; the suppression of deterministic chaos in dynamics of dissipative systems by noise (thoroughly discussed, *e.g.*, in [3]); synchronization by common noise [4]; Anderson localization [5]; the analog of the latter phenomenon for dissipative systems [6]; *etc.* In this Letter we disclose a novel phenomenon, where *weak* noise reduces disorder in chaotic dynamics. The effect is rooted in inhomogeneity of the distribution of natural measure over the strange attractor and impact of noise on this distribution.

First, we demonstrate this inhomogeneity and reason for the consequences expected from it. Then we construct an analytical theory for the noise impact. Finally, the predictions of the analytical theory are underpinned with results of simulation for paradigmatic chaotic system, the Lorenz one [9], where one can see the disorder reduction effect with divers quantifiers of regularity, and for the mapping which is relevant for one of the classical routes of the transition from regular dynamics to chaotic one [8].

In our treatment we relay on the idea of representation of chaotic attractor by the set of UPOs embedded into it ([10,11], *etc.*). On this way, for instance, one can evaluate the average $\overline{A}$ for the chaotic regime as $\overline{A} = \int A(\mathbf{x})\mu(\mathbf{x})d\mathbf{x}$, where μ is the natural measure, and integration is performed over the whole attractor. It was shown in [10]

(see also the text book [11]) that the distribution of natural measure over attractor can be recovered from properties of the UPOs. For two-dimensional maps (and therefore three-dimensional dynamic systems where one can use Poincaré section to construct such a map) the natural measure of an UPO is $\mu \propto \Lambda^{-1}$ where Λ is the largest multiplier of perturbations of this UPO. This result are treated to be valid in higher dimensions as long as we have only one positive Lyapunov exponent. The approach was employed already in [12] for evaluation of fractal characteristics of the Lorenz attractor from the UPOs data weighted with measure $\mu \propto \Lambda^{-1}$.

Inhomogeneity of natural measure distribution. – We consider the Lorenz system [9], as an example,

$$\frac{dx}{dt} = \sigma(y - x), \quad \frac{dy}{dt} = rx - y - xz, \quad \frac{dz}{dt} = xy - bz. \quad (1)$$

Recently [13], we reported the results of calculation of all UPOs with length $N \le 20$ (by "lenght" of UPO we mean the number of loops). In Fig.1 one can see probability density function (PDF) of $\langle z \rangle$ calculated over UPOs of length $N = 20$, the distribution weighted with Λ^{-1} and PDF of $\langle z \rangle$ calculated over 20-loop segments of the chaotic trajectory. The unweighted and weighted distributions are remarkably dissimilar and the latter matches the chaotic averaging much better. The weighted distribution possesses a relatively big tail for large $\langle z \rangle$, where there is a quite few number of UPOs (unweighted PDF nearly riches zero next to $\langle z \rangle = 24$). The origin of this difference is a huge dispersion of multipliers for long UPOs:
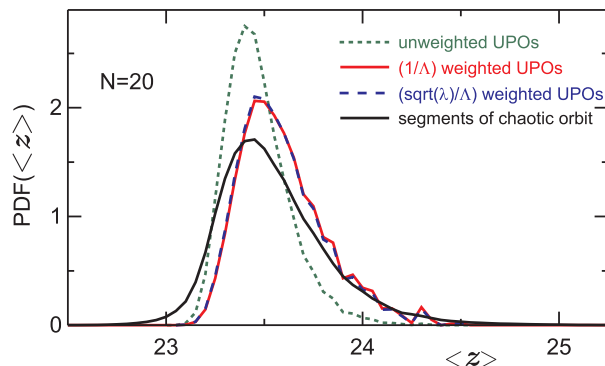
Fig. 1: (Color online) Probability density function of $\langle z \rangle$ calculated for the Lorenz system with classical set of parameters ($\sigma = 10$, $b = 8/3$, $r = 28$) over UPOs of length $N = 20$ without weight (doted green line), with weight Λ^{-1} (solid red line), which corresponds to UPO's measure within chaotic attractor in noise-free case, and with weight $\sqrt{\lambda}/\Lambda$ (dashed blue line), which corresponds to the maximal correction of the weight induced by a weak noise, and 20-loop segments of chaotic orbit (solid black line).

for $N = 20$ the smallest multiplier $\Lambda = 62\,200$ is by factor 72.8 smaller than the largest one $\Lambda = 4\,527\,160$. Noteworthy, the weighted distribution is significantly broader than the unweighted one. This feature implies a large contribution of relatively small number of UPOs with extreme deviation of properties from the typical ones (*e.g.*, in Fig. 1 these UPOs have untypically large values of $\langle z \rangle$). In terms of temporal evolution, the system dynamics is strongly inhomogeneous: the epoches of "typical" behaviour intermit with the epoches spent near several least unstable "untypical" periodic orbits.

Such a huge inhomogeneity of multipliers evokes the questions: (i) How this inhomogeneity is effected by weak noise? (ii) What are the consequences for the system dynamics? Indeed, the set of UPOs is dense and even infinitely small perturbation throws the system from one UPO onto another. Hence, one can expect that a very weak noise could first influence the rules of walking over this set and only then affect properties of UPOs. Presumably, the noise should play a "smoothing" role, reducing inhomogeneity of multipliers. As a result of such smoothing, one should observe, in particular, shrinking of the distribution of $\langle z \rangle$ calculated over finite segments of the chaotic trajectory because unweighted distribution is more narrow than the weighted one. One could also expect other kinds of regularization of the chaotic dynamics by weak noise.

Effect of noise on natural measure distribution. – A significant advance in analytical description of noise action on systems with fine structure was made for one-dimensional maps [14, 15]. In [14] the formalism of Feynman diagrams was employed for treatment of noisy dynamics as a perturbation to deterministic dynamics over the set of periodic orbits. In [15] the escape
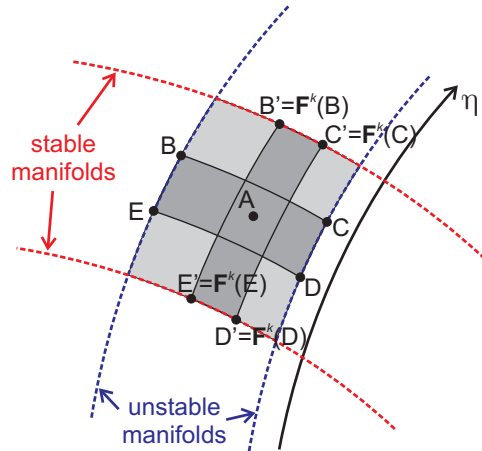

Fig. 2: (Color online) Cell on the Poincaré surface. In the vicinity of fixed point A of map $\mathbf{F}^k$, curvilinear parallelogram B'C'D'E' is the image of BCDE.

rates for unimodal maps were evaluated. However, in higher dimensions—which are practically inevitable when the map is a Poincaré map of chaotic continuous-time dynamic system—analytical advance becomes restricted; studies require *ad hoc* approaches and/or more numerics [7]. Moreover, for most problems addressed in literature, low-length periodic orbits provide enough data for an accurate description [14, 15], whereas in our case inhomogeneity of high-length orbits is essential.

For evaluation of natural measure, we first briefly recall its calculation for the no-noise case (interested readers can find a detailed rigorous consideration in Sec. VI of [10]). We consider certain Poincaré section and two-dimensional map on it, which is induced by three-dimensional phase flow. We are interested in calculation of the natural measure distribution over attractor on the Poincaré surface. Let us consider k iterations of the map, $\mathbf{x}_{n+k} = \mathbf{F}^k(\mathbf{x}_n)$, where k is large, and a small cell bounded by stable and unstable manifolds, which feature the directions of compaction and stretching of the phase volume, respectively, as sketched in Fig. 2. In the vicinity of fixed point A of map $\mathbf{F}^k$, curvilinear parallelogram B'C'D'E' is formed by states belonging the cell before k iterations of the map, and BCDE is the pre-image of this parallelogram. By definition of the ergodic (mixing) attractor, for any two subsets S_a and S_b in the phase space of the system, we have

$$\lim_{k \to \infty} \mu[S_a \bigcap \mathbf{F}^k(S_b)] = \mu(S_a)\,\mu(S_b).$$

If we take S_a and S_b to be the cell we consider, then $\mu(\text{cell}) = \mu(\text{BCDE})/\mu(\text{cell}) = \mu(\text{B'C'D'E'})/\mu(\text{cell})$. Furthermore, the measure varies smoothly along the unstable direction [16] (this is quite general in nature, that the stretching/expansion is a smoothing process), while it typically has a complex inhomogeneous structure in the stable direction, which is transversal to the chaotic attractor. Therefore, one can easily compare the measure of the

cell and parallelogram BCDE: their ratio is the ratio of their extension along the unstable direction, because these two sets have one and the same structure and extension along the "complex" inhomogeneous stable direction. We find $\mu(\text{cell}) = \mu(\text{BCDE})/\mu(\text{cell}) = \text{CD}/\text{C}'\text{D}' = 1/\Lambda_{\mathbf{F}^k}(\text{A})$, where $\Lambda_{\mathbf{F}^k}(\text{A})$ is the largest eigen value of fixed point A of map $\mathbf{F}^k$. Recall, that we have found the measure associated with a single fixed point A; when there is more fixed points in the cell one has to sum contributions of all of them,

$$\mu(\text{cell}) = \lim_{k \to \infty} \sum_{\text{fixed points in cell}} \frac{1}{\Lambda_{\mathbf{F}^k}(A_j)}.$$

Henceforth, the subscript j indicates the number of a fixed point or the corresponding UPO. This result was derived in [10]. Recall, the presented evaluation implies that attractor is hyperbolic, *i.e.*, the stable and unstable manifolds are not mutually tangent anywhere. Although the Lorenz attractor is non-hyperbolic, it is generally accepted that statistically significant measure distribution is determined by the same law $1/\Lambda$.

This calculation of the natural measure sheds light on the fact that we have to track only the coordinate along the unstable manifolds, say η, and evaluate the fraction of states in the ε-vicinity of some UPO, for which η stays within the ε-vicinity. In no-noise case, the fraction of such states is simply $1/\Lambda$. Now let us evaluate such a fraction in the presence of a weak Gaussian δ-correlated noise $a\xi(t)$: $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = 2\delta(t-t')$, a is the noise amplitude.

Our map is induced by a continuous time evolution and in the presence of time-dependent signal, noise, one has to deal with a continuous-time evolution. Placing the origin of the η-axis at fixed point A_j, one can find

$$
\begin{aligned}
\eta(t_0 + T_j) &= \Lambda_j \eta(t_0) \\
&\quad + a \int_0^{T_j} f[\mathbf{x}_j(\tau)]\, \xi(t_0 + \tau)\, e^{\lambda_j(T_j - \tau)} d\tau \\
&= \Lambda_j \eta(t_0) + \alpha R\,, \quad\quad\quad (2)
\end{aligned}
$$

where T_j is the period of UPO $\mathbf{x}_j(t)$ associated with A_j, $f(\mathbf{x})$ is the susceptibility to noise, $\lambda_j = T_j^{-1} \ln \Lambda_j$ is the leading Lyapunov exponent of the j-th UPO, R is the Gaussian random value of unit variance, $\langle R^2 \rangle = 1$, and

$$
\begin{aligned}
\alpha^2 &\equiv a^2 \left\langle \left(\int_0^{T_j} f[\mathbf{x}_j(\tau)]\, \xi(t_0 + \tau)\, e^{\lambda_j(T_j - \tau)} d\tau \right)^2 \right\rangle \\
&= 2a^2 \int_0^{T_j} f^2[\mathbf{x}_j(\tau)]\, e^{2\lambda_j(T_j - \tau)} d\tau = \frac{a^2 F_j^2 (\Lambda_j^2 - 1)}{\lambda_j}.
\end{aligned}
$$

Here F_j is the characteristic value of $f[\mathbf{x}_j(\tau)]$; $F_j^2 \equiv 2\lambda_j (1 - \Lambda_j^{-2})^{-1} \int_0^{T_j} f^2[\mathbf{x}_j(\tau)] \exp(-2\lambda_j \tau) d\tau$. According to Eq. (2), the probability of $\eta(t_0 + T_j)$ to be at the ε_j-vicinity

of fixed point A_j is

$$
\begin{aligned}
P(|\eta(t_0 + T_j)| \leq \varepsilon_j \,|\, \eta(t_0)) &= \int_{\frac{-\varepsilon_j - \Lambda_j \eta(t_0)}{\alpha}}^{\frac{\varepsilon_j - \Lambda_j \eta(t_0)}{\alpha}} \frac{e^{-\frac{R^2}{2}}}{\sqrt{2\pi}} dR \\
&= \frac{1}{2} \left[\Phi\left(\frac{\varepsilon_j - \Lambda_j \eta(t_0)}{\alpha\sqrt{2}} \right) - \Phi\left(\frac{-\varepsilon_j - \Lambda_j \eta(t_0)}{\alpha\sqrt{2}} \right) \right],
\end{aligned}
$$

where $\Phi(...)$ is the error function. Finally, the measure of the ε_j-vicinity of fixed point A_j, say $\Omega_{\varepsilon_j}(A_j)$, is

$$
\begin{aligned}
\mu[\Omega_{\varepsilon_j}(A_j)] &= \int_{-\varepsilon_j}^{\varepsilon_j} P(|\eta(t_0 + T_j)| \leq \varepsilon_j \,|\, \eta(t_0))\, d(\eta(t_0)) \\
&\approx \frac{1}{\Lambda_j} \Phi\left(\frac{\varepsilon_j}{F_j a} \sqrt{\frac{\lambda_j}{2}} \right). \quad\quad (3)
\end{aligned}
$$

The last approximate expression corresponds to the limit of large Λ_j, which is relevant for long UPOs we consider. For vanishing noise, $a = 0$, the error function turns 1, and we have the conventional result of [10].

One can see in Eq. (3), that for larger Λ_j (hence, larger λ_j) the argument of Φ is larger. This decreases inequivalence between UPOs with different weight. Maximal decrease of inequivalence is achieved when the argument of Φ is small, $(\varepsilon_j/F_j)\sqrt{\lambda/2} \ll a$; at this case one finds $\mu \propto \sqrt{\lambda_j}/\Lambda_j$. In particular, for the 20-loop UPOs in the Lorenz system the maximal ratio of weights decreases by 15%. In Fig. 1 the distribution of $\langle z \rangle$ with weight $\sqrt{\lambda}/\Lambda$ is slightly shrunk compared to the one with Λ^{-1}.

Notice, the vicinity extension ε is not precisely known value in our analytical theory. The properties of neighboring UPOs could be similar and, additionally, the measure smoothly varies along the unstable direction on the attractor. Hence, ε could be not only the characteristic distance between nearest UPOs of length N but rather a considerably larger number featuring the size of clusters of UPOs.

Effect of noise on disorder quantifiers. – Now we calculate the variance of $\langle z \rangle$ for UPOs of length N in the presence of weak noise $a\xi(t)$:

$$\text{var}(\langle z \rangle_j) = \sum_j \mu_j \left[\langle z \rangle_j^2 - \overline{\langle z \rangle}^2 \right] + K a^2, \quad \overline{\langle z \rangle} = \sum_j \mu_j \langle z \rangle_j. \quad (4)$$

Here we sum the variance related to the distribution over the set of UPOs of the noise-free system and the noise-induced distortion of these UPOs themselves, which is nearly statistically independent from the former and described by term $K a^2$. The value of parameter K is approximately inferred from numerical calculation of the variance of $\langle z \rangle$ for the segments of the chaotic trajectory (rough analytical estimation yields the value of the same order of magnitude).

In Fig. 3, we plot the variance of $\langle z \rangle$ for UPOs of length $N = 20$ for the Lorenz system (1) with the classical set of parameters. We approximately assume ε_j/F_j to be the
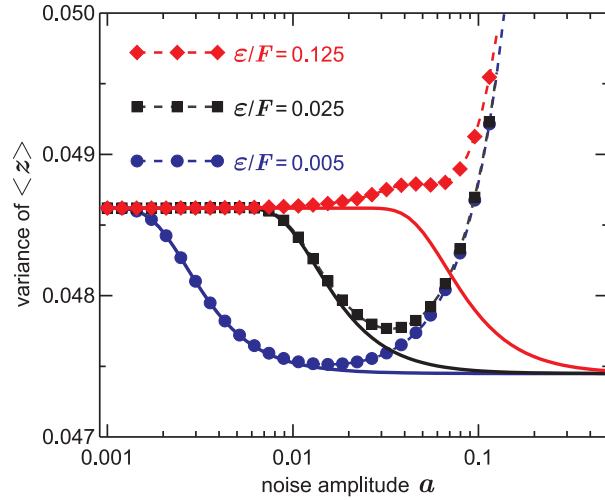
Fig. 3: (Color online) Theoretical dependence of the variance of $\langle z \rangle$ for UPOs of length $N = 20$ on noise amplitude a for the Lorenz system with classical set of parameters and various values of parameter ε which measures characteristic vicinity of UPOs. For the sake of demonstration we plot the first term of the expression for the variance [Eq.(4)] (solid lines) which features purely noise-induced redistribution of measure over UPOs of the noise-free system; the dashed lines with symbols represent the total variance with account for the noise-induced dispersion of each single UPO.
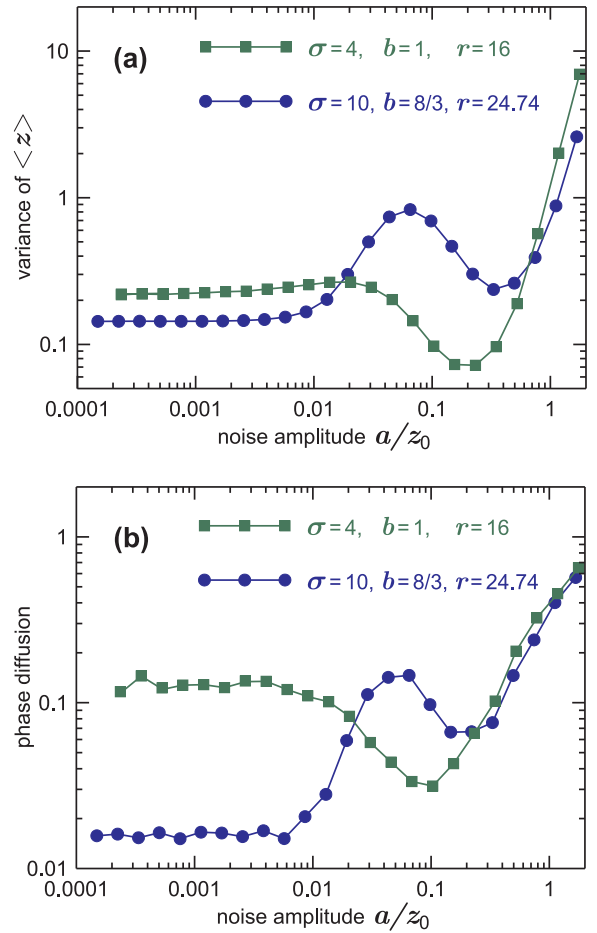

Fig. 4: (Color online) Noise-induced regularization of chaotic dynamics in the Lorenz system at $(\sigma = 1, b = 8/3, r = 24.74)$ and $(\sigma = 4, b = 1, r = 16)$: (a) variance of $\langle z \rangle$ calculated over 20-loop segments of the chaotic orbit and (b) phase diffusion coefficient measuring coherence of chaotic oscillations. Noise amplitude a in graphs is scaled to the coordinate of the saddle points $z_0 = r - 1$.

same for all UPOs (notice, F_j is of order of 1). One can note the always-present ordering role of noise which decreases inequality of weights of UPOs: solid lines show decrease of the variance by 2.5%. However, this ordering effect can be overwhelmed by the dispersive action of noise on the orbits when ε is not small enough. For the classical set of parameters the effect is not enough well pronounced and practically unobservable with dispersion of $\langle z \rangle$ for chaotic segments. Still, we report the results for this case, where the effect is small, for to emphasize their universality: the effect is always present in some form. For a stronger inhomogeneity of the attractor one can expect a more pronounced effect.

We performed simulations for the Lorenz system (1) with additive noise $a\xi(t)$ put in dz/dt. With the classical values of parameters $\sigma = 10$ and $b = 8/3$, the inhomogeneity of the chaotic set is strongest near the threshold where it becomes attracting, $r \approx 24.06$ (e.g., see [17, 18]). At $r \approx 24.74$ the chaotic attractor becomes the only attractor in the phase space (between the stability threshold and this point, it coexists with the stable fixed points which significantly influence the noise-perturbed dynamics of the system). In Fig. 4(a), the dependence of var$\langle z \rangle$ for 20-loop segments of the chaotic trajectory on the noise amplitude exhibits a well-pronounced minimum and resembles the theoretical dependencies in Fig.3. The effect is well pronounced though this local minimum corresponds to larger dispersion than in the noise-free case. In reality, the level of noise cannot be zero and this local minimum can pro-

vide a smaller dispersion than that for the minimal noise level if, for instance, noise cannot be diminished below $a = 0.05 z_0$ ($z_0 = r - 1$ is the coordinate of the saddle points). Furthermore, the local minimum can become a global one for a larger inhomogeneity of the attractor. In Fig. 4(a), the dependence of var$\langle z \rangle$ on the noise amplitude for $\sigma = 4$, $b = 1$, $r = 16$ (where the chaotic attractor nearly touches the slightly unstable non-trivial saddle points) has the minimum which provides 3 times smaller dispersion than the noise-free case.

The dispersion of $\langle z \rangle$ calculated over finite segments of the chaotic trajectory is only a sample of a quantifier with which one can observe the ordering effect of noise we report. This ordering can be observed with other quantifiers of the system dynamics as well. For instance, one of the very important characteristics of the chaotic systems is the coherence of their oscillations. In particular, coherence determines susceptibility of the system to control forcing and
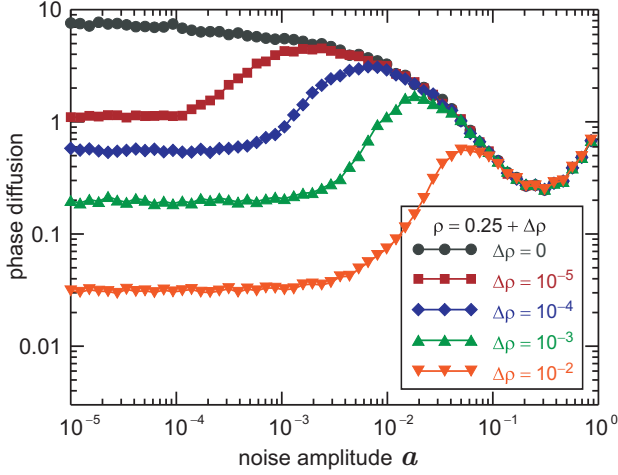
Fig. 5: (Color online) Noise-induced regularization of chaotic dynamics for mapping (5).

predisposition to synchronization (*e.g.*, see [19, 20]). The coherence is quantified by the diffusion coefficient of the chaotic oscillation phase (the oscillation phase to be introduced so that it growths by 2π for one revolution of the trajectory). In Fig. 4(b) the dependencies of the phase diffusion coefficient on the noise amplitude are plotted for the same sets of parameters as we used for calculation of the dispersion of $\langle z \rangle$. One can see well pronounced minima. For $\sigma = 4$, $b = 1$, $r = 16$ the diffusion can be suppressed by factor 5 in comparison with the noise-free case.

Observation of the noise-induced regularization effect with mapping. – Our analytical treatment is not restricted to the Lorenz system which is utilized only for demonstration. The Lorenz system is known to possess significant peculiarities [18]; therefore it would be appropriate to validate the effect we observe with independent examples. The classical scenario of the transition to chaos in the Lorenz system is known to be a special case of the transition to chaos via cascade of bifurcations of homoclinic loops [8]. Within the vicinity of the bifurcation accumulation point, a general system, where such a scenario of the transition to chaos occurs, can be described by the following mapping (in the noise-free case) [8];

$$x_{n+1} = \begin{cases} x^\nu - \rho_1, & \text{for } x_n > 0; \\ -A_2|x|^\nu + \rho_2, & \text{for } x_n < 0; \end{cases}$$

$$t_{n+1} = t_n + 1 + \begin{cases} T_1 \ln(\rho_1/x_n), & \text{for } x_n > 0; \\ T_2 \ln(\rho_2/|x_n|), & \text{for } x_n < 0; \end{cases}$$

where x_n is the system state at the time moment t_n, ν is the saddle index, ρ_i are the parameters featuring deviation from the bifurcation accumulation point ($\rho_1 = \rho_2 = 0$ at this point). The iteration $n \to n+1$ corresponds to the $(n+1)$-th revolution of the system—the oscillation phase grows for 2π as $n \to n+1$. For the sake of definiteness we restrict ourselves to the case of $\rho_1 = \rho_2 \equiv \rho$, $\nu = 1/2$,

$A_2 = 1$, $T_1 = T_2 = 1$ and introduce noise:

$$x_{n+1} = \text{signum}(x_n)(\sqrt{|x_n|} - \rho) + a\zeta_n;$$
$$t_{n+1} = t_n + 1 + \ln(\rho/|x_n|). \tag{5}$$

Here a is the noise amplitude and ζ_n are independent random numbers uniformly distributed in $\left[-\sqrt{3}, \sqrt{3}\right]$ (*i.e.*, $\langle \zeta^2 \rangle = 1$).

For this mapping the phase grows uniformly with each iteration while time increments $(t_{n+1} - t_n)$ are nonconstant. The phase diffusion can be calculated from t_n as a function of n (see, *e.g.*, [21]). In Fig. 5 one can see that the regularizing effect of noise can be as well observed near $\rho = 0.25$ where the natural measure distribution is highly inhomogeneous.

Conclusion. – We have reported the ordering effect of noise on the chaotic dynamics. This effect is rooted into inhomogeneity of the natural measure over the attractor and the fact that noise diminishes this inhomogeneity. Without noise, the measure is determined by the largest multipliers of unstable periodic orbits (UPOs) composing the chaotic attractor and is typically extremely nonuniform for long orbits due to the exponential dependence of the multiplier on the orbit length and the Lyapunov exponent which vary for different UPOs. Generally, the unweighted probability density function of some average for UPOs is centered around the value which does not have to correspond to the maximal measure (see Fig. 1 where the big tail of the weighted PDF clearly indicates that the natural measure of UPOs with large $\langle z \rangle$ is drastically larger than that of UPOs near the peak of the unweighted PDF). Hence, the dynamics is more contributed by UPOs which are not "geometrically typical". Noise smooths the measure over the set of UPOs and reduces the role of excursions along the "untypical" UPOs. On these grounds, we expect the effect to be a universal feature for chaotic dynamics with exception for some model systems with absolutely uniform natural measure (like the tent map). With strong enough inhomogeneity given, the ordering effect can overwhelm the distorting action of noise on UPOs and lead to significant observable ordering of the system dynamics. We have developed the analytical theory for the effect of noise on the measure distribution over chaotic attractor and observed the ordering effects for dispersion of averages over finite segments of the chaotic trajectory and coherence of chaotic oscillations. Noteworthy, the reported phenomenon is novel; its mechanism being robust and vital is not related to any of well-known phenomena of noise-induced ordering (stochastic [1] or coherence resonance [2], suppression of deterministic chaos by noise [3], *etc.*).

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